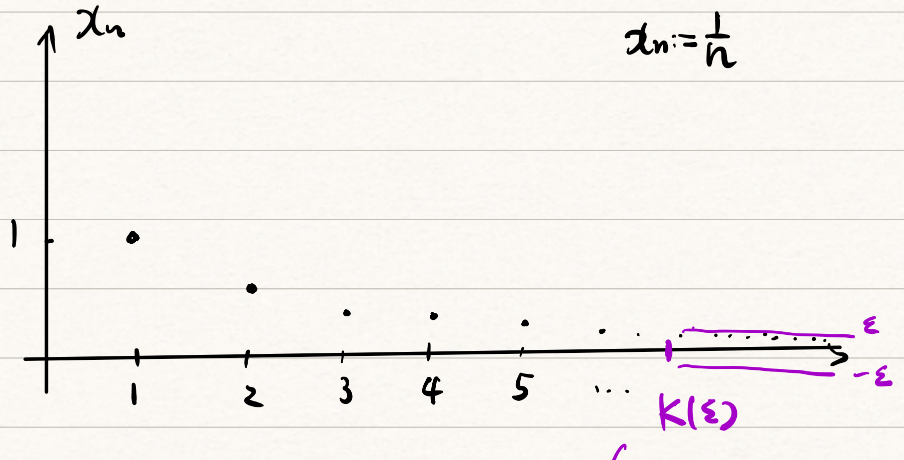
# Limit of a Sequence

Definition: Let $X := (x_n)$ be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$ be a real number. $X$ is said to converge to $x$ if for every $\varepsilon > 0$, there exists a natural number $N$ s.t

$$|x_n - x| < \varepsilon, \quad \forall\, n \geq N.$$

Then $x$ is called the limit of $(x_n)$.

Property: if limit exists, the it's unique.




$|x_n - 0| < \varepsilon$ for $n \geq K(\varepsilon)$.

How to prove $\lim_{n\to\infty} x_n = x \in \mathbb{R}$ ?

① Let $\varepsilon > 0$ be given.

② Find a useful estimate for $|x_n - x|$.

③ Find $K(\varepsilon) \in \mathbb{N}$ s.t. the estimate in ② is less than $\varepsilon$ whenever $n \geq K(\varepsilon)$.

④ Complete the argument.

Exercise:

1. Prove $\lim_{n\to\infty}\left(\frac{n^2-n}{2n^2+3}\right) = \frac{1}{2}$.

Proof:

① Let $\varepsilon > 0$ be given

② Note that for $n \geq 1$,

$$\left| \frac{n^2-n}{2n^2+3} - \frac{1}{2} \right| = \left| \frac{(2n^2-2n)-(2n^2+3)}{2(2n^2+3)} \right|$$

$$= \frac{2n+3}{2(2n^2+3)} \leq \frac{2n+3}{n^2} \leq \frac{2n+3n}{n^2} = \frac{5}{n}.$$

③ Let $K := \lfloor \frac{5}{\varepsilon} \rfloor + 1$

④ Then, for all $n \geq K$, we have

$\frac{5}{n} < \varepsilon$

$\Uparrow$

$n > \frac{5}{\varepsilon}$

$$\left| \frac{n^2-n}{2n^2+3} - \frac{1}{2} \right| \leq \frac{5}{n} \leq \frac{5}{K} < \varepsilon.$$

Therefore, $\lim_{n\to\infty} \frac{n^2-n}{2n^2+3} = \frac{1}{2}$.

$\square$

<u>Example 2</u>: Prove $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = 0$.

Proof: ① Let $\varepsilon>0$ be given.

② Note that, for $n\geq 1$,

$$\sqrt{n+1}-\sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{2\sqrt{n}}.$$

$a^2-b^2$
$=(a+b)(a-b)$

③ Let $K := \lfloor \frac{1}{4\varepsilon^2} \rfloor + 1$.

$\frac{1}{2\sqrt{n}} < \varepsilon$
$\Uparrow$
$n > \frac{1}{4\varepsilon^2}$

④ Then, for any $n\geq K$, we have

$$\left|(\sqrt{n+1}-\sqrt{n})-0\right| \leq \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{K}} < \varepsilon.$$

Therefore, by definition,

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = 0.$$

$\square$

How to prove a sequence is divergent?

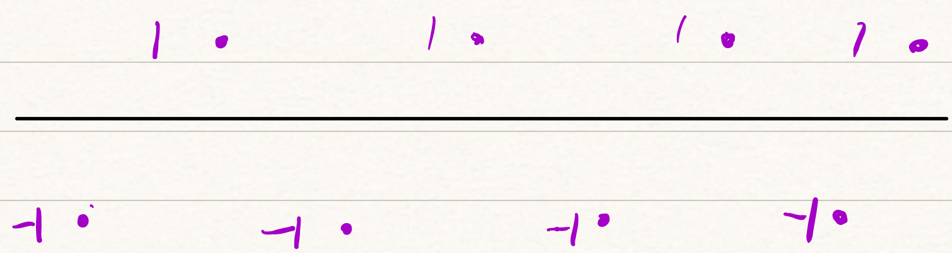
Or disprove $\lim_{n\to\infty} x_n = x$.

There exists some $\varepsilon_0 > 0$ s.t. $\forall N$, $\exists x_n$ with $n > N$

s.t. $|x_n - x| \geq \varepsilon_0$.

Equivalently, $\exists$ subsequence $x_{n_j}$ s.t. $|x_{n_j} - x| \geq \varepsilon_0$.

Example:



Example 3:

Show that the sequence $(-1, 1, -1, 1, \ldots)$ is divergent.

$$\left( x_n = \begin{cases} -1, & \text{if } n \text{ is odd.} \\ 1, & \text{if } n \text{ is even.} \end{cases} \right)$$

Need to prove $\forall x \in \mathbb{R}$. $(x_n)$ does not converge to $x$.

Choose $\varepsilon_0 = 1$.

If $x \leq 0$, then we can choose $|x_{2n} - x| = 1 - x \geq 1 = \varepsilon_0$

If $x > 0$, then choose $|x_{2n+1} - x| = x + 1 > 1 = \varepsilon_0$.

In any case, for any large $N$,

there exists some $n > N$ s.t. $|x_n - x| \geq \varepsilon_0$.

Therefore, the sequence $(x_n)$ is divergent.

$\square$